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ON STABILITY OF MOTION RELATIVE TO A PART OF THE VARIABLES FOR LINEAR SYSTEMS WITH CONSTANT OR ALMOST-CONSTANT MATRICES*

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Conditions for stability and asymptotic stability relative to a part of the variables are examined for the motion of linear systems. Criteria for stability and asymptotic stability of motion relative to a part of the variables have been established for systems with constant coefficients. Sufficient conditions for stability and asymptotic stability of motion relative to a part of the variables are derived for systems with almost-constant coefficients. The paper succeeds /1—3/.

1. We consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = A\mathbf{x} \tag{1.1}$$

in which A is an nth-order constant square matrix, $\mathbf{x} \in E_n$. We represent vector $\mathbf{x}$ in the form /1,2/

$$\mathbf{x} = (y_1, \ldots, y_m, z_1, \ldots, z_p) = (\mathbf{y}, \mathbf{z})$$
$$m > 0, \quad p \geqslant 0, \quad m + p = n$$

The stability of the unperturbed motion $\mathbf{x} = 0$ relative to variables $y_1, \ldots, y_m$ will be called $\mathbf{y}$-stability. If we introduce the $m \times n$-matrix

$$H = \begin{Vmatrix} 1 & 0 \ldots 0 & 0 \ldots 0 \\ 0 & 1 \ldots 0 & 0 \ldots 0 \\ \cdot & \cdot \cdot \cdot \cdot & \cdot \cdot \cdot \cdot \\ 0 & 0 \ldots 1 & 0 \ldots 0 \end{Vmatrix}$$

then vector $\mathbf{y}$ can be presented as $\mathbf{y} = H\mathbf{x}$. Conditions for the asymptotic $\mathbf{y}$-stability of motion $\mathbf{x} = 0$ of system (1.1) are given in /3/ wherein the asymptotic stability of motion $\mathbf{x} = 0$ relative to a part of the variables was investigated for the system $\mathbf{x}^{\cdot} = A\mathbf{x} + \boldsymbol{\varphi}(t, \mathbf{x})$. This result is presented in Theorem 2. At first we prove an auxiliary assertion whose n-dimensional analog can be found in /4/.

Lemma. The motion $\mathbf{x} = 0$ of system $\mathbf{x}^{\cdot} = A(t)\mathbf{x}$ with a matrix $A(t)$ piecewise-continuous on $[0, \infty]$ is:

1) $\mathbf{y}$-stable if and only if the component $\mathbf{y}(t)$ of each solution $\mathbf{x}(t)$ is bounded on $[0, \infty)$;

2) asymptotically $\mathbf{y}$-stable if and only if the component $\mathbf{y}(t)$ of each solution $\mathbf{x}(t)$ tends to zero as $t \to \infty$.

Proof. 1) Let the motion $\mathbf{x} = 0$ be $\mathbf{y}$-stable. For arbitrary $\varepsilon > 0$, $t_0 \geqslant 0$ we can find $\delta(\varepsilon, t_0) > 0$ such that for any solutions $\mathbf{x}(t)$, from $\|\mathbf{x}(t_0)\| < \delta$ follows $\|\mathbf{y}(t)\| < \varepsilon$ when $t \geqslant t_0$. Let us consider the fundamental system of solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ satisfying the conditions $\|\mathbf{x}_i(t_0)\| < \delta$, $i = 1, \ldots, n$. The fundamental matrix $\Phi(t) = [\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]$ set up from these solutions admits of the bound $\|H\Phi(t)\| \leqslant L$ for $t \geqslant 0$, where $L > 0$. Consequently, for each solution $\mathbf{x}(t)$ we have

$$\|\mathbf{y}(t)\| = \|H\mathbf{x}(t)\| = \|H\Phi(t)\mathbf{c}\| \leqslant L\|\mathbf{c}\|$$

when $t \geqslant 0$. Conversely, let the component $\mathbf{y}(t)$ of each solution $\mathbf{x}(t)$ be bounded. Let us consider the fundamental matrix $\Phi(t)$ satisfying the condition $\Phi(t_0) = I$, where I is the unit matrix. There exists $L > 0$ such that $\|H\Phi(t)\| \leqslant L$ when $t \geqslant 0$; therefore, from $\mathbf{x}(t) = \Phi(t)\mathbf{x}(t_0)$ follows $\|\mathbf{y}(t)\| = \|H\Phi(t)\mathbf{x}(t_0)\| \leqslant L\|\mathbf{x}(t_0)\|$. If now we choose $\delta = \varepsilon L^{-1}$, then from $\|\mathbf{x}(t_0)\| < \delta$ follows $\|\mathbf{y}(t)\| < \varepsilon$ for $t \geqslant t_0$.

2) Let the motion $\mathbf{x} = 0$ be asymptotically $\mathbf{y}$-stable. There exists $\Delta(t_0) > 0$ such that each solution $\mathbf{x}(t)$ for which $\|\mathbf{x}(t_0)\| < \Delta$ satisfies the condition

$$\lim \|\mathbf{y}(t)\| = 0, \quad t \to \infty \tag{1.2}$$

The fundamental matrix $\Phi(t)$ set up from the solutions $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ satisfying the conditions $\|\mathbf{x}_i(t_0)\| < \Delta (i = 1, \ldots, n)$ possesses the property $\|H\Phi(t)\| \to 0$ as $t \to \infty$. Therefore, for each solution $\mathbf{x}(t)$ we have

$$\lim \|\mathbf{y}(t)\| = \lim \|H\Phi(t)\mathbf{c}\| = 0, \quad t \to \infty$$

Conversely, let the component $\mathbf{y}(t)$ of each solution $\mathbf{x}(t)$ satisfy condition (1.2). Then

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$\|H\Phi(t)\| \to 0$ as $t \to \infty$, where $\Phi(t)$ is the fundamental matrix satisfying the condition $\Phi(t_0) = I$. Consequently, $\|H\Phi(t)\| \leqslant L, t \geqslant 0$, for some $L > 0$. From $\|y(t)\| \leqslant L\|x(t_0)\|$ if follows that if we choose $\delta = \varepsilon L^{-1}$, then $\|y(t)\| < \varepsilon$ for $t \geqslant t_0$, which together with (1.2) signifies the asymptotic y-stability of motion $x = 0$.

Root vectors /5/ play an essential role in what follows. Let $\lambda_1, \ldots, \lambda_k$ be pairwise-distinct eigenvalues of matrix A and let s_α linearly-independent eigenvectors in space E_n correspond to the eigenvalue λ_α. We denote these vectors $v_{11}^\alpha, \ldots, v_{s_\alpha 1}^\alpha$; $v_{\beta\gamma}^\alpha$ is a root vector of height γ, generated by the eigenvector $v_{\beta 1}^\alpha$. Having multiplied the matrix $\exp[(A - \lambda_\alpha I)t]$ by the root vector $v_{\beta\gamma}^\alpha$, we obtain the relation

$$\exp[(A - \lambda_\alpha I)t]\, v_{\beta\gamma}^\alpha = v_{\beta\gamma}^\alpha + t(A - \lambda_\alpha I)\, v_{\beta\gamma}^\alpha + \ldots + \frac{t^{\gamma-1}}{(\gamma-1)!}(A - \lambda_\alpha I)^{\gamma-1} v_{\beta\gamma}^\alpha$$

whence follows the equality

$$\exp(At)\, v_{\beta\gamma}^\alpha = \exp(\lambda_\alpha t)\,(v_{\beta\gamma}^\alpha + t\, v_{\beta\gamma-1}^\alpha + \ldots + \frac{t^{\gamma-1}}{(\gamma-1)!} v_{\beta 1}^\alpha) \tag{1.3}$$

In the proper subspace corresponding to eigenvalue λ_i we can choose the basis $v_{11}^i, \ldots, v_{s_i 1}^i$ such that to each eigenvector v_{r1}^i $(r = 1, \ldots, s_i)$ there corresponds /5/ the root vectors $v_{r1}^i, \ldots, v_{rp_r}^i$ defined by the relations

$$\begin{aligned} A v_{r1}^i &= \lambda_i v_{r1}^i \\ A v_{r2}^i &= \lambda_i v_{r2}^i + v_{r1}^i \\ &\ldots\ldots\ldots\ldots \\ A v_{rp_r}^i &= \lambda_i v_{rp_r}^i + v_{rp_r-1}^i \end{aligned}$$

where these root vectors $(r = 1, \ldots, s_i;\ i = 1, \ldots, k)$ form a basis in space E_n.

Let us consider an arbitrary solution $x(t) = \exp(At)x_0$ of system (1.1). We expand vector x_0 with respect to the base root vectors of matrix A

$$x_0 = \sum_{i=1}^{k} \sum_{r=1}^{s_i} (b_{r1}^i v_{r1}^i + \ldots + b_{rp_r}^i v_{rp_r}^i)$$

By (1.3) we obtain

$$\exp(At)\, x_0 = \sum_{i=1}^{k} \sum_{r=1}^{s_i} \exp(\lambda_i t)\,(b_{r1}^i v_{r1}^i + b_{r2}^i (v_{r2}^i + t v_{r1}^i) + \ldots + b_{rp_r}^i (v_{rp_r}^i + t v_{rp_r-1}^i + \ldots + \frac{t^{p_r-1}}{(p_r-1)!} v_{r1}^i))$$

We pick out the component $y(t)$ of this solution

$$y(t) = Hx(t) = \sum_{i=1}^{k} S_i(t)$$

$$S_i(t) = \sum_{r=1}^{s_i} \exp(\lambda_i t)\left(b_{r1}^i H v_{r1}^i + b_{r2}^i (H v_{r2}^i + tH v_{r1}^i) + \ldots + b_{rp_r}^i \left(H v_{rp_r}^i + tH v_{rp_r-1}^i + \ldots + \frac{t^{p_r-1}}{(p_r-1)!} H v_{r1}^i\right)\right)$$

We separate the index set $\Omega = \{1, \ldots, k\}$ into three subsets $\Omega_1, \Omega_2, \Omega_3$. In them we include indices $i \in \Omega$ for which $\operatorname{Re}\lambda_i = 0$, $\operatorname{Re}\lambda_i < 0$, $\operatorname{Re}\lambda_i > 0$, respectively. Then vector $y(t)$ can be presented as

$$y(t) = \sum_{i\in\Omega_1} S_i(t) + \sum_{i\in\Omega_2} S_i(t) + \sum_{i\in\Omega_3} S_i(t) \tag{1.4}$$

Theorem 1. Let $\operatorname{Re}\lambda_i = 0$ for $i = 1, \ldots, l$ and $\operatorname{Re}\lambda_i \neq 0$ for $l = l+1, \ldots, k$. Motion $x = 0$ of system (1.1) is y-stable if and only if subspace $G = \{x: Hx = 0\}$ contains:

a) the root vectors corresponding to eigenvalues λ with $\operatorname{Re}\lambda = 0$, except, perhaps, the vectors of maximum height, i.e., the vectors $v_{r1}^i, \ldots, v_{rp_r-1}^i$ $(r = 1, \ldots, s_i;\ i = 1, \ldots, l)$;

b) the root vectors corresponding to eigenvalues λ with $\operatorname{Re}\lambda > 0$.

Proof. We first prove the sufficiency of the theorem's conditions. If both conditions are fulfilled, then (1.4) takes the form

$$y(t) = \sum_{i\in\Omega_1} \sum_{r=1}^{s_i} \exp(\lambda_i t)\, b_{rp_r}^i H v_{rp_r}^i + \sum_{i\in\Omega_2} S_i(t)$$

Since $\operatorname{Re}\lambda_i = 0$ when $i \in \Omega_1$ and $\operatorname{Re}\lambda_i < 0$ when $i \in \Omega_2$, all summands of the expression obtained are bounded on $[0, \infty)$. Consequently, there exists $L > 0$ such that $\|y(t)\| \leqslant L$ for $0 \leqslant t < \infty$.

Applying the lemma, we obtain the $\mathbf{y}$-stability of motion $\mathbf{x} = 0$.

Let us prove the necessity of the theorem's conditions. Assume that the first condition is violated. Suppose that for some $i \in \Omega_1$ and for some r $(1 \leqslant r \leqslant s_i)$ we find a number $q \geqslant 1$ such that

$$H\mathbf{v}^i_{rp_r-q} \neq 0 \tag{1.5}$$

In such case we take vector $\mathbf{v}^i_{rp_r}$ as $\mathbf{x}_0$. Then

$$\mathbf{y}(t) = H\exp(At)\,\mathbf{v}^i_{rp_r} = \exp(\lambda_i t)\left(H\mathbf{v}^i_{rp_r} + \ldots + \frac{t^q}{q!}H\mathbf{v}^i_{rp_r-q} + \ldots + \frac{t^{p_r-1}}{(p_r-1)!}H\mathbf{v}_{r1}{}^i\right) \tag{1.6}$$

Hence by virtue of (1.5) it follows that $\|\mathbf{y}(t)\|$ is not bounded as $t \to \infty$. Now assume that the theorem's second condition is violated. Suppose that for some $i \in \Omega_2$ and for some r we find a number $q \geqslant 0$ such that

$$H\mathbf{v}^i_{rp_r-q} \neq 0 \tag{1.7}$$

As in the preceding case we take vector $\mathbf{v}^i_{rp_r}$ as $\mathbf{x}_0$. Then on the basis of equality (1.6) and condition (1.7) we obtain the unboundedness of $\|\mathbf{y}(t)\|$ as $t \to \infty$.

Theorem 2. Motion $\mathbf{x} = 0$ of system (1.1) is asymptotically $\mathbf{y}$-stable if and only if all root vectors of matrix A corresponding to eigenvalues λ with $\operatorname{Re}\lambda \geqslant 0$ belong to subspace $G = \{\mathbf{x} : H\mathbf{x} = 0\}$.

We can convince ourselves of the validity of this statement by using relations (1.4) and (1.6). An algebraic variant of the theorem has been presented in /3/.

2. The next two theorems are an extension of Bellman's results to the case of stability relative to a part of the variables. We consider the system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = (A + B(t))\,\mathbf{x} \tag{2.1}$$

where matrix $B(t)$ is piecewise-continuous on $[0, \infty)$.

Theorem 3. If motion $\mathbf{x} = 0$ of system (1.1) is $\mathbf{y}$-stable, then so is the motion $\mathbf{x} = 0$ of system (2.1) under the condition that the last $n - m$ columns of matrix $B(t)$ consist of zeros and

$$\int_0^\infty \|B(t)\|\,dt < \infty$$

Proof. By the Cauchy formula we have

$$\mathbf{x}(t) = \exp(At)\,\mathbf{x}_0 + \int_0^t \exp[A(t-\tau)]\,B(\tau)\,\mathbf{x}(\tau)\,d\tau$$

Having multiplied this equality by H, we obtain

$$H\mathbf{x}(t) = H\exp(At)\,\mathbf{x}_0 + \int_0^t H\exp[A(t-\tau)]\,B(\tau)\,\mathbf{x}(\tau)\,d\tau \tag{2.2}$$

Since motion $\mathbf{x} = 0$ of system (1.1) is $\mathbf{y}$-stable, the function $H\exp(At)$ is bounded, i.e. $\|H\exp(At)\| \leqslant M$ for $0 \leqslant t < \infty$. Therefore, the inequality

$$\|H\mathbf{x}(t)\| \leqslant M\|\mathbf{x}_0\| + \int_0^t M\|B(\tau)\|\,\|H\mathbf{x}(\tau)\|\,d\tau$$

is valid, whence

$$\|H\mathbf{x}(t)\| \leqslant M\|\mathbf{x}_0\|\exp\left[M\int_0^t \|B(\tau)\|\,d\tau\right] \leqslant M\|\mathbf{x}_0\|\exp\left[M\int_0^\infty \|B(\tau)\|\,d\tau\right]$$

This estimation signifies that the component $\mathbf{y}(t)$ of each solution $\mathbf{x}(t)$ of system (2.1) is bounded. Consequently, by the lemma the motion $\mathbf{x} = 0$ is $\mathbf{y}$-stable.

If we consider the system

$$y^{\cdot} = (1 + t^2)^{-1}z, \quad z^{\cdot} = z$$

we can discover that, in general, we cannot waive the requirement that the last $n - m$ columns of matrix $B(t)$ be zero. Indeed, if we set

$$A = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & (1 + t^2)^{-1} \\ 0 & 0 \end{Vmatrix}$$

the hypotheses of Theorem 3 are fulfilled except for the requirement that the last $n-m=1$ column of matrix B be zero. Each solution of the system has the form

$$y(t)=y_0+z_0\int_0^t(1+\tau^2)^{-1}\exp(\tau)\,d\tau,\qquad z(t)=z_0\exp(t)$$

Whence if follows that if $z_0\neq 0$ the quantity $\|y(t)\|$ is not bounded as $t\to\infty$.

Theorem 4. If motion $\mathbf{x}=0$ of system (1.1) is asymptotically $\mathbf{y}$-stable, then so is the motion $\mathbf{x}=0$ of system (2.1) under the condition that the last $n-m$ columns of matrix $B(t)$ consist of zeros and $\|B(t)\|\leqslant\varepsilon$, where ε is sufficiently small.

Proof. We set $2a=\max_{i\in\Omega_2}\operatorname{Re}\lambda_i$. Then

$$\|H\exp(At)\|=\Big\|\sum_{i\in\Omega_2}\exp(\lambda_i t)\,Q_i(t)\Big\|\leqslant\sum_{i\in\Omega_2}\|Q_i(t)\|\exp(2at)\qquad(2.3)$$

Here $Q_i(t)$ are polynomials of degree no higher than n; therefore, the estimation of (2.3) can be replaced by

$$\|H\exp(At)\|\leqslant c\exp(at)$$

In such case, from (2.2) we obtain

$$\|H\mathbf{x}(t)\|\leqslant c\|\mathbf{x}_0\|\exp(at)+c\varepsilon\int_0^t\exp[a(t-\tau)]\,\|H\mathbf{x}(\tau)\|\,d\tau$$

After multiplying both sides of this inequality by $\exp(-at)$ and applying the Gronwall—Bellman inequality, we obtain the estimation

$$\|H\mathbf{x}(t)\|\exp(-at)\leqslant c\|\mathbf{x}_0\|\exp(c\varepsilon t)$$

If $c\varepsilon+a<0$, the relation

$$\lim\|H\mathbf{x}(t)\|=0\quad\text{as}\quad t\to\infty$$

is fulfilled, which, by the lemma, is equivalent to the asymptotic y-stability of motion $\mathbf{x}=0$ of system (2.1).

The example

$$y^{\cdot}=-y+\varepsilon(1+|t|^{-1})_z,\quad z^{\cdot}=z$$

shows that here too we cannot waive the requirement that the last $n-m$ columns of matrix $B(t)$ be zero. Indeed, if we set

$$A=\begin{Vmatrix}-1&0\\0&1\end{Vmatrix},\quad B=\begin{Vmatrix}0&\varepsilon(1+|t|)^{-1}\\0&0\end{Vmatrix}$$

then the conditions of Theorem 4 are fulfilled except for that requirement. For any $\varepsilon\neq 0$ the component

$$y(t)=\exp(-t)\Big(y_0+\varepsilon\int_0^t(1+|\tau|)^{-1}\exp(2\tau)\,d\tau\Big)$$

is not bounded as $t\to\infty$.

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